Lecture No. 29

Measure and Integration

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$f(x)$ $g(x)$

$\chi_A(x)$ $\chi_B(y)$

$$f : X \times Y \longrightarrow \mathbb{R}$$

<u>Assume</u> (i) $f \in L_1(X \times Y)$

<u>To Show</u> (ii) $\displaystyle\int_Y \left(\int_X |f(x,y)| \, d\mu(x) \right) d\nu(y) < +\infty$?

(i) $\Rightarrow$ $\displaystyle\int_{X \times Y} |f(x,y)| \, d\mu \times \nu < +\infty$

$|f(x,y)|$ is a non-negative mble function on $X \times Y$

F. Thm - I

$\Longrightarrow$

$$\int_Y \left(\int_X |f(x,y)|\, d\mu(x) \right) d\nu(y)$$

$$= \int_{X \times Y} |f(x,y)|\, d(\mu \times \nu) < +\infty$$

(i) $\Longrightarrow$ (ii)

(ii) $$\int_Y \left(\int_X |f(x,y)|\, d\mu(x) \right) d\nu(y) < +\infty$$

$|f(x,y)|$ is a non-negative meas fn.

F. Thm I

$$\Longrightarrow \int_{X\times Y} |f(x,y)|\, d(\mu\times\nu) = \int_Y \left(\int_X |f(x,y)|\, d\mu(x) \right) d\nu(y) < +\infty$$

$\Rightarrow$ (i) holds: $f \in L_1$

$X = [0,1]$ | $Y = [0,1]$

$\mathcal{A} = \mathcal{B}_{[0,1]}$ | $\mathcal{B} = \mathcal{B}_{[0,1]}$

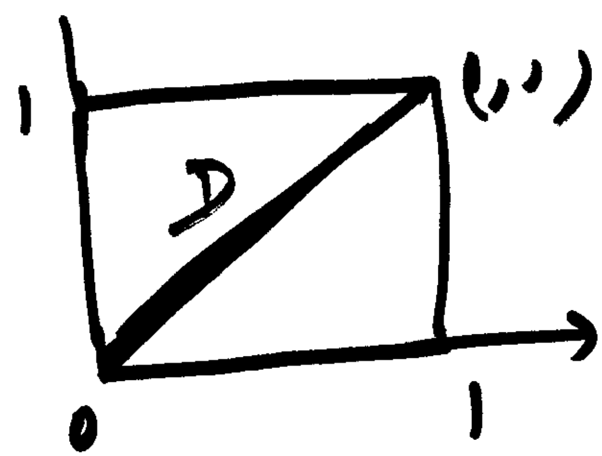
μ = Lebesgue Measure | ν = Counting Measure

$$(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$$

$D \subseteq X \times Y$

$D = \{(x,y) \mid x = y\}$

D is closed in $X \times Y$ ||



$\{(x_n, y_n)\}_{n\geq 1} \in D, \quad (x_n, y_n) \longrightarrow (x, y)$

$\Downarrow$

Claim $x = y.$

$x_n \longrightarrow x$

$y_n \longrightarrow y.$

Note $x_n = y_n$

$\Longrightarrow \quad x = y.$

$\Longrightarrow \quad D \in \mathcal{A} \otimes \mathcal{B}.$

$$\int_X \chi_D(x,y)\, d\mu(x) = 0 \quad \forall y \in Y?$$

Fixed y

$$\chi_D(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

$$\Rightarrow \int_X \chi_D(x,y)\, d\mu(x) = \int_X \chi_{\{y\}}(x)\, d\mu(x) = 0$$

<u>Fixed</u> x, $\chi_D(x,y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$

$$\int_Y \chi_D(x,y)\, d\nu(y) = 1 \quad \forall x \in X.$$

$f : ~~X~~\ [0,1] \times [0,1] \longrightarrow \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

f is continuous on $X \times Y$ except at $(0,0)$,

$\Rightarrow$ f is cont a.e.

$\Rightarrow$ f is Borel measurable. ||

$(X, \mathcal{A}, \mu)$

$f : X \longrightarrow \mathbb{R}$

$f \in L_1(X)$

$(Y, \mathcal{B}, \nu)$

$g : Y \longrightarrow \mathbb{R}$

$g \in L_1(Y)$

$$\Phi : X \times Y \longrightarrow \mathbb{R}$$

$$\Phi(x,y) = f(x)\, g(y) \quad \forall (x,y)$$

<u>Claim</u> $\Phi \in L_1(X \times Y)$.

i.e. $\displaystyle\int_{X \times Y} |\Phi(x,y)| \, d(\mu \times \nu) < +\infty$??

$\Rightarrow$

Enough to show

$$\int_X \left(\int_Y |\Phi(x,y)| \, d\nu(y) \right) d\mu(x) < +\infty \, ?$$

$$\parallel$$

$$\int_X \left(\int_Y |f(x)| \, |g(y)| \, d\nu(y) \right) d\mu(x)$$

$$= \int_X |f(x)| \left(\int_Y |g(y)| \, d\nu(y) \right) d\mu(x)$$

$$= \left(\int_X |f(x)|\, d\mu(x) \right) \left(\int_Y |g(y)|\, d\nu(y) \right)$$

$$< +\infty$$

$$\Longrightarrow \quad \phi \in L_1(X \times Y)$$

$$\xRightarrow{\text{FTh 2}} \int_{X\times Y} \phi(x,y)\, d\mu\times\nu = \int_X \left(\int_Y f(x)\, g(y)\, d\nu \right) d\mu(x)$$

$$= \int_X f(x) \left(\int_Y g\, d\nu \right) d\mu$$

$$= \left(\int_X f(x)\, d\mu(x) \right) \left(\int_Y g(y)\, d\nu(y) \right)''$$